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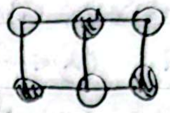
Chap 1 • Dimer models

(1) defined on bipartite graphs : 

(2) we can do a perfect matching to the graph : a subset of edges without common vertices, touch every vertices exactly once. 

(3) we do weight (according to model) sum of perfect matching to get the partition function of the dimer model

$$Z = \sum_{\substack{\text{perfect} \\ \text{matching}}} (\text{weight of this matching})$$

Since dimer models and free fermions in string theory both exactly solvable and free fermions are omnipresent, we search dimer models then.

(4) it is found : 2d Young Diagram $\xleftrightarrow[\text{1-1}]{\text{Maya Diagram}}$ 1-d dimer models

3d Young Diagram $\xleftrightarrow[\text{honeycomb}]{}$ 2-d dimer models

(~~by~~ we mean perfect matching here)

However, 3d Young diagram is related to

- crystal melting picture of topological A model on non-compact toric CY
- Nekrasov partition func.

Dimer models also relate to 4d N=1 quiver gauge theory, where bipartite graphs on T^2, named brane tilling.

Chap 2 Quiver gauge theory from toric CY cone

(1) Quivers : orientated graph, consists nodes and arrows.

node: SU(N) (U(N)), note: Sp, SO appears in orientifolded brane tilings.

~~quiver~~ arrow: bifundamentals between two gauge groups.

We restrict to 4d $N \geq 1$ afterwards.

note: specifing the SUSY requires superpotential fixed.

(gauge)
(2) Anomaly Cancellation.

label vertices in $a, b, \dots$, arrows in $I, J, \cdots$, define

$$\sigma(I,a) = \begin{cases} 1 & \textcircled{a} \xleftarrow{I} \\ -1 & \textcircled{a} \xrightarrow{I} \end{cases}$$

<u>Thm</u> (Cancellation of node a): $\sum_{I=(a,b)} \sigma(I,a) N_b = 0$

for all node has same rank, this means #In = # Out.

(3) D3 brane engineering

<u>Thm</u> (McKay Correspondence)

Discrete subgroup $\Gamma \subset SU(2) \leftrightarrow$ $\overset{\text{semi}}{\text{simple}}$ simple-laced Lie alg.

<u>Pf</u>: (1) for $\Gamma \subset SU(2)$, all irrep $\{\rho_i\}$, associate each ρ_i to a node in Dynkin diagram;

then, $\rho_r \otimes \rho_i = \bigoplus a^{ij} \rho_j$ reproduce the Dynkin Diagram, for a^{ij} arrows from ρ_i to ρ_j; #

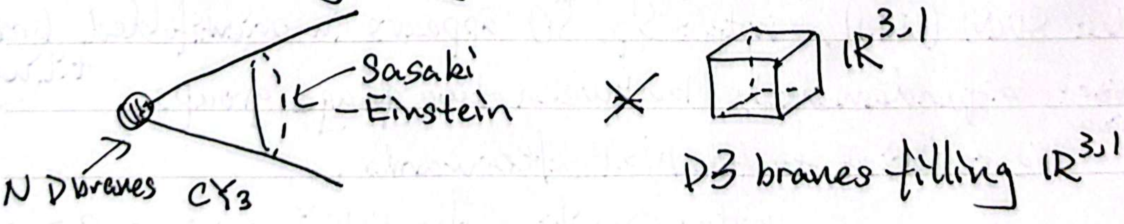
(2) minimal resolution of $\mathbb{C}/\Gamma$ has several exceptional divisors, dual of the configuration results a quiver, also $\underset{\text{the same}}{\text{Dynkin}}$ #

Physically, consider N D branes on ALE (resolution of $\mathbb{C}^2/\Gamma$), the worldvolume susy is the quiver gauge theory. Interestingly the moduli of such quiver gauge theory reproduce ALE! ~~Genera~~

- This generally has $N=2$ for $\Gamma \subset SU(2)$
  $N=1$ for $\Gamma \subset SU(3)$

(4) toric CY's relation to D brane engeeving.

On a cone-type singularity, we <u>"probe"</u> D3 brane:



these toric CY are non-compact, worldvolume theory gravity decouples, results a worldvolume gauge theory, whose moduli is the CY cone this gauge theory is believed to be a quiver gauge theory.

Toric Diagram	$\xrightarrow{\text{inverse-problem}}$ / $\xleftarrow{\text{forward-problem}}$	Quiver Diagram
1	to	many

toric duality (related to Seiberg duality)

brane tiling ~~solves~~ gives the inverse and forward algorithm.

Chap 3 Brane Tiling as a five brane system.

1. Construction.
2. Strong coupling limit.

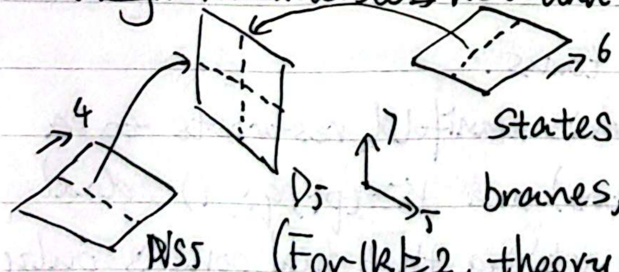
◦ construct 5 brane system that represents a $\mathcal{N}=1$ SCFT

1. IIB theory with N D5 branes, results a worldvolume $U(N)$ gauge theory
2. compactify 2 directions on T^2 with R as radius. (~~of D5 brane~~) take x^5, x^7 of D^5 brane.
3. divide D5 by NS5 into different regions.

8,9	0,1,2,3		4	5	6	7
	•	D5		•		•
	•	NS5	•	•		
	•	NS5'			•	•

D5 and NS5 (NS5') intersect at T^2 1-cycles, ~~to~~ NS5 and NS5' does not intersect.

~~Although No Intersects~~ NS5 and D5 juction  exists.



Some regions of T^2 becomes bound states of N D5-branes and k NS5-branes, which is called as (N,k) brane. (For $|k|\geq 2$, theory is not conventional quiver gauge theory, we don't discuss). These T^2 regions are ~~constructed~~ divided by NS5 cycles, these cycles encodes NS5 charge, we now give rules of imposing these cycles.

1° on T^2, cycles are represented by arrows across N $\uparrow$ N+h

2° $\vec{\alpha}_\mu$ as this arrow $=(p_\mu, q_\mu)$ (α and β winding), we have

$$\sum p_\mu = \sum q_\mu = 0 \iff \sum_\mu \alpha_\mu = 0$$

this picture is actually all NS5 brane becomes one at the junction area. Assuming worldvolume $\mathbb{R}^{3,1}\times\Sigma$, we have

	0	1	2	3	4	5	6	7	8	9
NS5	✓					Σ				
D5	✓					○		○		

where $\Sigma\cap T^2$ has thoses cycles.

(2) from D5 to D3.

T dual the T^2, $D5\Rightarrow D3$, $NS5\Rightarrow$ backgraund geometry CY.

Tdual Recall:

1. at worldsheet level, Tdual: $\partial_\sigma X\Leftrightarrow\partial_\tau X$, this makes ~~one~~ the dual direction disappear for D brane. $X(\tau,\sigma)=x_0+\alpha'(\frac{n}{R})\tau+wR\sigma$

$R\Rightarrow\frac{\alpha'}{R}$: $n\Leftrightarrow w$, $\sigma\Leftrightarrow\tau$

2. Buscher Rules: t-dual at y

$$\begin{cases} G'_{yy}=\frac{1}{G_{yy}} \\ G'_{\mu y}=\frac{B_{\mu y}}{G_{yy}} \quad B'_{\mu y}=\frac{G_{\mu y}}{G_{yy}} \\ G'_{\mu\nu}=G_{\mu\nu}-\frac{G_{\mu y}G_{\nu y}-B_{\mu y}B_{\nu y}}{G_{yy}} \quad B'_{\mu\nu}=B_{\mu\nu}-\frac{G_{\mu y}B_{\nu y}-B_{\mu y}G_{\nu y}}{G_{yy}} \\ \Phi'=\Phi-\frac{1}{2}\ln(G_{yy}) \end{cases}$$

3. NS5 as magnetic source of $B_{\mu\nu}$

$$F1\to\underset{2}{B_{\mu\nu}}\to\underset{3}{dB}\longrightarrow\underset{7}{*dB}\longrightarrow\underset{6}{\tilde{B}}\ \text{on NS5}$$

In this way we come back to the D3 probe picture.

Techniques

1. Specifying NS5 cycles $\vec{\alpha}_\mu$ on torus.

recall that CY condition of toric manifold restricts to a hypersurface, 1d cones (vectors) are $v_i=(p_i,q_i,1)$, dual cone defined by $v_i\cdot w\geq 0\ \forall i$, noticing that dual cone is only decided by v_α at the corner, each correspond to a facet in the dual cone note as F_α, this is the web-diagram.

We can regard toric CY as T^3 fibred over dual cone. On facet the T^3 shrinks to T^2: $(\phi_1,\phi_2,\phi_3)=\theta_1(1,0,0)+\theta_2(0,1,0)+\theta_3(p_\alpha,q_\alpha,1)$

$\theta_1=\phi_1-p_\alpha\phi_3 \quad \theta_2=\phi_2-q_\alpha\phi_3 \quad \theta_3=\phi_3$

Buscher in form language

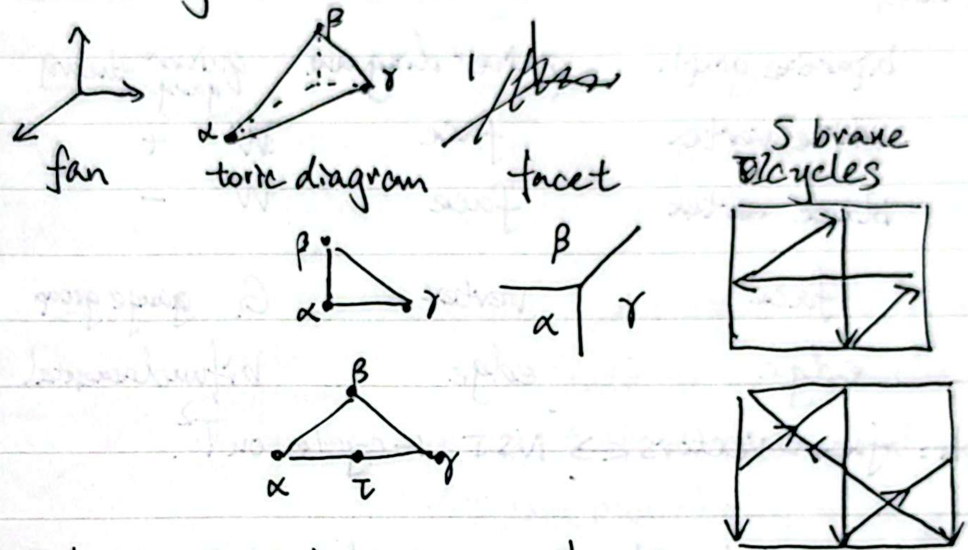
$$ds_{10}^2 = ds_9^2 + g_{99}(dx^9+v_1)^2 \quad \begin{cases} x^9 \to x^9+\lambda \\ v_1 \to v_1 - d\lambda \end{cases}$$

B field: $B = b_2 + b_1\wedge(dx^9+v_1) \xrightarrow[\text{T-dual}]{} B' = b_2' + b_1'\wedge(dx^9+v_1')$

$b_2' = b_2 + b_1\wedge v_1$

$b_1' = v_1 \quad v_1' = b_1$

NS5 cycles in ~~NS5~~ brane picture is mapped to toric diagram in 3 brane picture.



(3) from 5 brane to quiver gauge theory

G: for (N,0) area, correspond to gauge group SU(N)

but for (N,±1) area, correspond to U(1), not dynamical.

Fields: for (N,0) intersection point, we have massless open strings as bifundamentals. Due to NS5 brane, all these ~~fd~~ bifund. are chiral.

Thm let $\Gamma \subset SU(3)$, the D3 picture with toric ~~quiver~~ CY $\mathbb{C}^3/\Gamma$ has the same quiver as gauge theory with McKay Quiver by Γ.

2.

5 brane picture also gives superpotential in quiver gauge theory, which is given by $\sum_k h_k \operatorname{tr}\prod_{I\in k}\Phi_I$, h_k is take to be $\begin{cases} +1 & k:(N,1) \\ -1 & k:(N,-1) \end{cases}$

The F term equation of this superpotential is the defining equation of Calabi-Yan, as long as we use brane tiling we can only ~~describe~~ describe toric CY or its deformation.

(4) 5 brane and bipartite

Rules from 5 brane to bipartite graph:

1. white vertex: $(N,1)$ black: $(N,-1)$

2. connect two vertex if their polygon intersects.

this also explains quiver diagram as the dual of bipartite (on Torus)

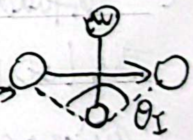
Sum up: pictures

five brane	bipartite graph	quiver diagram	quiver gauge theory	~~toric graph~~
$(N,1)$ brane	white vertex	face	W +	
$(N,-1)$ brane	black vertex	face	W −	
$(N,0)$ brane	face	vertex	G gauge group	
open string	edge	edge	bifundamental	

toric graph: facet vectors $\Leftrightarrow$ NS5 1-cycle on T^2.

Also it is able to get 5 brane picture from bipartite graphs by zigzag pathes: start with one edge and one direction, turn all left (right) at white (black) vertex, all possible zigzag paths 1-1 to 1 cycles in 5 brane picture.

One proposal to include R charge in bipartite picture is by imposing isoradially embedding, then $R_I = \theta_I/\pi$ where



node of quiver

for $\sum_{I\in k}\theta_I = 2\pi$ $\sum_{I\in a}(\pi-\theta_I) = 2\pi \Rightarrow \underbrace{\sum_{I\in k} R_I = 2}_{}$, $\underbrace{\sum_{I\in a}(1-R_I) = 2}_{}$

Marginality of superpotential

Vanishing of NSVZ β function

2. Others.

(1) string coupling.

strong coupling limit: $g_s \to \infty$, $l_s \to 0$, $R \to 0$ $\quad \frac{R^2}{g_s l^2}$ fixed.

weak ~ : $g_s \to 0$, $l_s \to 0$, $R \to 0$

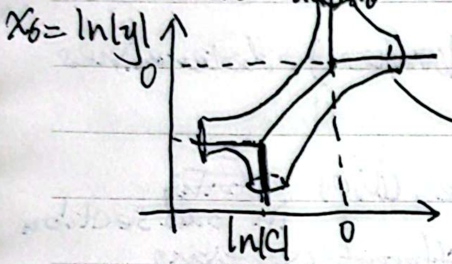
$T_{D5} \sim \frac{1}{g_s l_s^6}$ $\quad T_{NS5} \sim \frac{1}{g_s^2 l_s^6}$ $\quad$ We mainly focus on weak ~.

where $T_{NS5} > T_{D5}$ and the surface Σ becomes a holomorphic curve.
We can write the explicit expression of Σ as the ~~zero~~ locus of

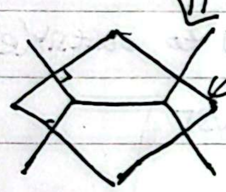
$$P(x,y)=\sum_{(k,l)\in\Delta\cap\mathbb{Z}^2} c_{k,l}\, x^k y^l \quad \text{where } s = x_4+ix_5 \quad x=e^{2\pi s}$$
$$t = x_6+ix_7 \quad t=e^{2\pi t}$$

Ex. conifold $P = c_{0,0}+c_{0,1}y+c_{1,0}x+c_{1,1}xy$ $\quad y=-\frac{x+c}{1+x}$

rescale x_4, x_6 $= c+x+y+xy$ four punctures: $(-1,\infty)$ $(-c,0)$ $(0,-c)$ $(\infty,-1)$



projection of NS5 on x_4, x_6 direction and its "amoeba" $\nearrow$ in $R\to\infty$ limit, amoeba becomes spines "tropical limit"



toric diagram of conifold.

: ~~only~~ for $(x,y)\to(r^m x, r^n y)$ dominate in $r\to\infty$ is $ml+nk$ maximize. only at (n,m) orthogonal to toric diagram can choice of k,l multivalues.

~~a no~~ another visualize of $P(x,y)$ is Coamoebae, which is $\hat{A}(P)=\mathrm{Arg}(P^{-1}(0))$, changing coefficients keeps its asymptotic boundary, which is determined by toric diagram. For coamoebae, its tropical limit gives exactly the bipartite diagram.

Chap4 More on brane-tiling

4.1 fractional branes : more general rank assignments.

weak coupling D5 charge conservation: $\sum_a N_a C_a = 0$ in $H_1(\bar{\Sigma},\mathbb{Z})$

Anomaly cancellation: $\sum G(b,a) N_b = \sum \underbrace{\langle C_b, C_a\rangle}_{\text{intersection number}} N_b = \sum (C_b N_b, C_a) = 0$

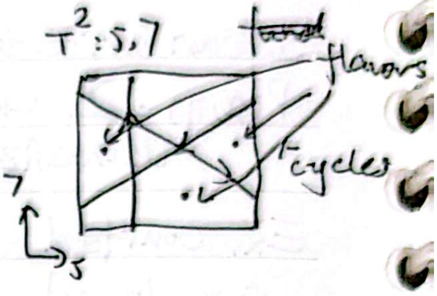
4.2 flavor branes: introducting ~~fermions~~ flavor quark.

We focus on flavor 5 brane on 4,6 directions. when the T duality at 5 and 7 direction is worked, this becomes D7 branes wrapped on divisors in toric CY.

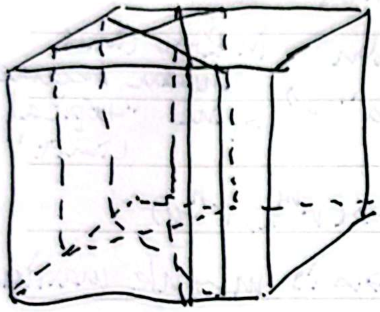
T^2 on 5,7

recall the 5 brane picture.

	0,1,2,3	4	5	6	7	8	9
D5	•		•		•		
NS5	•	•	•				
NS5'	•			•	•		
NS5	•			Σ			
D5f	•	•		•			



like a water tank, every section ((N,k) brane.) determines SU(N) (for (N,0))

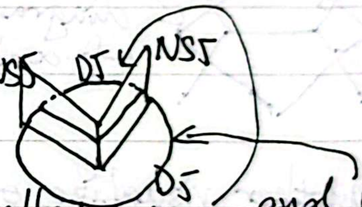


gloune: string ~~between~~ ~~these~~ (N,0) ~~string~~ in one section

bifund.: string connecting different (N,0) sections

flavor: string from one (N,0) to outside.

one subtle we mention is flavor D5 : NS5



where two possible flavor branes are called minor and major

4.3 BPS conditions.

One natural question ~~asking~~ for us is all the information we have used are topological: they are free from deformations. However realistic physics is not always topological, thus we should investigate what changes if we deform sth.: like the NS5 1-cycle.

We ~~first~~ consider BPS condition and moduli parameters for weak/strong coupling.

1. Weak coupling

recall $\Sigma = P(x^5, x^7)$ as a holomorphic curve at weak coupling.

denote $\Pi(x^5,x^7)$ as the 2d plane along 46 directions with fixed x^5, x^7 coord. $Q(x^5,x^7)$ is the discrete function takes value of the intersection number of $\Pi(x^5,x^7)$ and $P(x^5,x^7)$: Only if $|Q(x^5,x^7)|=0,1$ can we do the previous construction, BPS condition requires us to constrain the asymptotic behavior of Σ.

which is parametrized by $\begin{cases} M_\mu = \lim_\mu (n_\mu x^4 - m_\mu x^6) \\ \xi_\mu = \lim_\mu (n_\mu x^5 - m_\mu x^7 + \frac{1}{2}) \end{cases}$

where $(n_\mu, \boxed{m_\mu})$ is the normal vector of integral the μ-th edge.

M_μ describes external web lines on x^4-x^6

ξ_μ describes α_μ on the torus.

The BPS condition is proved to be $\sum M_\mu = \sum \xi_\mu = 0$;

- Adding D5 branes impose more constrains.

2. Strong coupling

When D5 on T^2 is almost flat, $M_\mu = 0$ and the configuration is completedly determined by ξ_μ;

4.4 Marginal deformation of quiver gauge theory

We choose all coefficient in superpotential to be ± 1, which is quite unnatural physically, we have g_a ($g_s + \theta_a$) and h_k as two perameters, and we want to find ~~exact~~ marginal deformation keeps $\beta_a = \beta_k = 0$ which is parametrized by g_a, h_k;

4.5 Kasteleyn matrix and fast forward algorithm.

- (height function) for an arbitrary choosed perfect matching D_0, height change for a another perfect matching D from D_0 is $h_1(D, D_0) = \langle D - D_0, \vec{\alpha} \rangle$ $h_2(D, D_0) = \langle D - D_0, \vec{\beta} \rangle$

the characteristic polynomial is $P(x,y) = \sum_D (-1)^{h_1(D)h_2(D) + h_1(D) + h_2(D)} x^{h_1(D)} y^{h_2(D)}$

this is not easy because we are supposed to know all D's.

- (Kasteleyn matrix)

first we have the adjacency matrix of bipartite graph, all

whites are W, blacks are B. $A_{bw} = \begin{cases} 1 & b, w \text{ is connected} \\ 0 & \text{not} \end{cases}$

$\mathrm{perm}(A) = \sum_{\sigma \in S_n} A_{1\sigma(1)} \cdots A_{n\sigma(n)}$ counts the # of perfect matching.

We modify every element by sign $B_{bw} = S_{bw} A_{bw}$

to let $\det B = \mathrm{perm}\, A = Z$

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this choice of sign is the Kasteleyn condition

1. for every $2m$ edges face of bipartite polygon, product of all signs is $(-1)^{m+1}$

2. we can always choose s_I to satisfy this condition

2. for T^2, the Kasteleyn matrix is defined by

$$K_{mo}(x,y) = \sum_I s_I A_I \, x^{\langle I,\alpha \rangle} y^{\langle I,\beta \rangle}$$

$\det K$ is called characteristic polynomial ← summing over all edges connecting b/w

∘ (Fast Forward Algorithm)

Bipartite graph ⟶ Kasteleyn Matrix ⟶ Characteristic Polynomial $P(x,y)$

↓

Toric Diagram ⟸ = Newton Polygon of $P(x,y)$

Def (Newton Polygon) For $P(x,y) = \sum_{(k,l)} c_{kl} x^k y^l$

$\Delta(P) := $ convex hull of $\{(k,l) \in \mathbb{Z}^2 \mid c_{kl} \neq 0\}$.

One can show the characteristic we defined by height function is the same with Kasteleyn matrix up to an $SL(2,\mathbb{Z})$ $\begin{pmatrix} x \\ y \end{pmatrix}$ transformation.

4.6 perfect matching and F-term constraints

a natural product between an edge and perfect matching is

$$\langle I, m_D \rangle = \begin{cases} 1 & I \in m_D \\ 0 & I \notin m_D \end{cases}$$

(Claim) for each perfect matching with complex variable p_D the composition $X_I = \prod_D p_D^{\langle I, m_D \rangle}$ is the solution of F-term constraints: $\frac{\partial W}{\partial X_I} = 0$

4.7 Seiberg Duality

Of Course, Seiberg Duality arise from SQCD, but it is natural to extend it to $N=1$ quiver gauge theories; for the case of SQCD

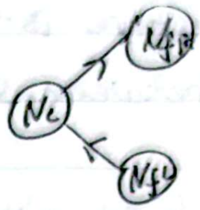
Eletric: $N=1$ $SU(N_c)$ N_f fund. Q_i N_f afund. $\bar{Q}^i$ conformal window $N_f \in [\frac{3}{2}N_c, 3N_c]$

Magnetic: $N=1$ $SU(N_f N_c)$ N_f fund q_i N_f afund $\bar{q}^i$ N_f^2 meson M^i_j

superpotential $W = q_i M^i_j \bar{q}^j$

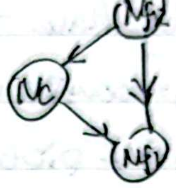
e:

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$
Q_i	$\square$	$\overline{\square}$	1
$\overline{Q}_i$	$\overline{\square}$	1	$\square$



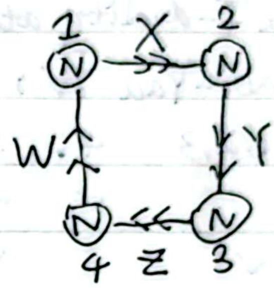
m:

	$SU(N_f-N_c)$	$SU(N_f)_L$	$SU(N_f)_R$
q_i	$\square$	$\overline{\square}$	1
$\overline{q}_i$	$\overline{\square}$	1	$\square$
M	1	$\square$	$\overline{\square}$



Ex. F_0 (one typical quiver) $i=1,2$

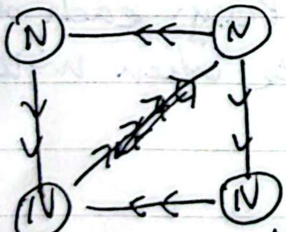
	$SU(N)_1$	$SU(N)_2$	$SU(N)_3$	$SU(N)_4$
X_i	$\square$	$\overline{\square}$		
Y_i		$\square$	$\overline{\square}$	
Z_i			$\square$	$\overline{\square}$
W_i	$\overline{\square}$			$\square$



$W = \varepsilon^{ij}\varepsilon^{kl}\,\mathrm{tr}(X_i Y_k Z_j W_l)$

We can do Seiberg Duality for a node, for node 1 as example, locally $N_f = 2N$

Doing Seiberg duality { exchange arrows on one node
for each arrow get a meson on the two flavor nodes



~~here~~

the dualed field content

	1	2	3	4
q_i	$\overline{\square}$	$\square$		
Y_i		$\square$	$\overline{\square}$	
Z_i			$\square$	$\overline{\square}$
$\overline{q}_i$	$\square$			$\overline{\square}$
M_{ij}	[illegible]	$\overline{\square}$		$\square$

and Superpotential is deformed

$W^{dual} = W + M_{ij} q_i \overline{q}_j$

where original $X_i Z_j$ is changed to M_{ij}

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one conjecture is that all quiver for one toric diagram are related between seiberg duality.

4.8 a look at mirror D6-brane picture.

We already have two pictures describing the same result (which is ofcourse related), namely the 5brane picture and the D3 brane picture. Another one is the mirror D6 brane picture.

Do a T-duality at 9 direction, $D5 \to D6$, $NS5 \to$ background Calabi-Yau W, and D6 brane cycles 3 cycle in W;

	0,1,2,3	4	5	6	7	8	9
NS5		✓	✓	✓	✓	✓	✓
D6	✓		✓		✓		✓

W is given by $P(x,y)=\cancel{W}\,uv$ $(x,y\in\mathbb{C}^*,\ u,v\in\mathbb{C})$ as the mirror of Calabi-Yau M. And 3 cycles are generated by the shape of $W=0$ to $W=W_*$, thus the number of 3cycles equals to the critical pts. of $P(x,y)$, or number of gauge groups.

Note the CY3 is actually a double fibration $\begin{cases} P(x,y)=W & (\Sigma_g) \\ uv=W & (\text{fibres}) \end{cases}$ which is for every pt. of W choice, uv and $P(x,y)$ each defines a fibre, and when $W=0$ the uv one degenerate when $W=W_*$ the $P(x,y)$ one degenerates.

One way to specify these 3 cycles is by introducing vanishing paths, which can naturally define Lagrangian submanifolds in these CYs.